Contracting in Delegated Portfolio Management: The Case of Alternative Assets

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Abstract

The typical portfolio management contract in traditional investments like mutual funds resembles a linear sharing rule with a management fee expressed as a proportion of assets under management. On the other hand, in the case of alternative investments like hedge funds and private equity funds, such contracts typically include an option-type incentive fee in addition to the proportional asset-based fee. The substantial payouts often resulting from such contracts have led critics to ask if the ‘two plus twenty’ contracts are in the clients’ best interests. Or are they simply evidence of the market power wielded by the alternative asset managers?

This paper explores conditions under which the first best outcome may be achieved via linear contracts in the delegated portfolio management setting. The analysis suggests that a benchmark-linked linear contract can achieve the first best outcome when (a) the manager’s portfolio allocation is not based on private information, and (b) the investor is able to noiselessly observe the asset returns. Next, the paper examines the limitations of linear contracts when the portfolio is invested in ‘opaque’ assets that are relatively illiquid or privately held. We show that the first best outcome is no longer feasible in this case and the second-best optimal contract features an option-type component. The relative importance of the option-type component is an increasing function of the portfolio’s opacity. Further, the principal’s utility loss from restricting the weight of the option-type component to zero is increasing in the asset’s opacity. These results help provide a rationale for the form of contracts observed in the case of alternative investments like hedge funds.
Contracting in Delegated Portfolio Management: The Case of Alternative Assets

There is a wide disparity in the kind of portfolio management contracts we observe in settings where the investor (principal) delegates the management of her assets to an investment manager (agent). For example, in the case of traditional investments including mutual funds, the typical contract resembles a linear sharing rule with an advisory fee expressed as a proportion of assets under management. On the other hand, in the case of alternative investments like hedge funds and private equity funds, such contracts typically include a performance-based incentive fee in addition to the proportional asset-based fee.\textsuperscript{1} Anecdotal evidence suggests that the proportional fee is between 1 and 2 percent of the assets under management while the incentive fee typically equals 20 percent of the profits. These fee arrangements have often resulted in substantial rewards for the top performing investment managers leading some to wonder whether the fees are in fact excessive. More generally, critics have wondered whether the ‘two plus twenty’ contracts typical in the case of the alternative asset management industry are in fact in the clients’ best interests? Or are they simply evidence of the market power wielded by the alternative asset managers that allows them to impose fee arrangements that are in their own best interests? While it is certainly plausible that existing fee arrangements may be due to a less than perfectly competitive industry structure, an important unresolved question is: What should the form of the compensation contract look like in the absence of such market imperfections? Given the growing importance of alternative assets in institutional portfolios, these questions have become increasingly relevant in recent years. Moreover, the issue has attracted considerable scrutiny in the aftermath of the recent financial crisis in 2008 along with calls for increased regulation of hedge funds.

The goal of this paper is twofold. One, it explores conditions under which linear contracts can be optimal in the delegated portfolio management setting. Two, it aims to explain the observed differences in the form of the portfolio management contracts based on the nature of the assets held in the underlying

\textsuperscript{1} In the case of mutual funds, current regulations restrict performance-linked incentive fees to be symmetric, i.e., incentives for over-performance should mirror the penalties for underperformance.
portfolio. As we discuss below, extant theoretical models have difficulty in explaining the differences in the kind of contracts observed across the various segments of the asset management industry. A key aspect of our analysis is that it does not rely on market imperfections such as imperfect competition in the hedge fund or private equity sector to explain the observed contract features. Our analysis relies on the difference in the composition of portfolios of hedge funds and private equity funds compared to traditional mutual funds. Hedge funds and private equity portfolios typically invest in relatively illiquid or privately held securities. By contrast, traditional mutual fund portfolios typically invest in publicly traded, relatively liquid securities. The liquid and transparent nature of such portfolios makes it feasible to implement suitable benchmarks to evaluate manager performance. As we subsequently show, the relative lack of liquidity and opacity of the alternative asset portfolios makes it considerably harder for an investor to benchmark their performance, making the principal-agent problem harder to resolve. Intuitively, with an increase in the difficulty of benchmarking performance, a more flexible form of contract becomes desirable.

We consider a prototypical delegated portfolio management problem in which the agent (the portfolio manager) manages a portfolio that is invested in a single risky asset and a risk free asset. We begin with a very general framework that allows us to identify conditions under which the first best outcome may be achieved via linear portfolio management contracts. We show that observability of asset returns is a critical feature that allows for linear contracts to be optimal in this setting. We further show that the general framework can accommodate a number of recent models in the literature and thus helps explain a number of recent results in this area. Equally important, the analysis helps identify the conditions that limit the usefulness of linear contracts in the more realistic delegated portfolio management environment.

As discussed above, a particular setting of interest is the case when the portfolio manager invests in illiquid or non-public assets, making the portfolio relatively ‘opaque’. In the second part of the analysis we examine how the asset opacity impacts the form of the contract. We model the degree of transparency or opacity of a portfolio of assets via the correlation of the portfolio’s returns with the appropriate
publicly observed benchmark.² We first show that the ability to implement the first best solution is critically dependent upon the adequacy of the available information regarding the returns of the assets in the portfolio in question. The implementation involves the use of a performance benchmark against which the portfolio manager is evaluated. The appropriate benchmark tracks the first best portfolio strategy. The ability to construct an accurate benchmark for this purpose is critically dependent on the investor’s ability to observe the returns of the assets in the portfolio. Intuitively, with perfect observability of asset returns, the investor can determine the terminal value of the (first best) benchmark portfolio and penalize any deviation from this benchmark value. This makes it feasible to enforce the first best outcome in this setting.

The above result naturally leads to the question: “What if the investor can only construct a noisy benchmark, i.e., a benchmark that tracks the first best strategy only imperfectly?” To answer this question we consider the case when the investor’s information about the returns of the underlying assets in the portfolio is less than perfect. This case is modeled by treating the correlation between the risky asset and the benchmark as being strictly less than one. We show that in this case the agent’s allocation to the actively managed risky asset is less than in the first best case. In other words, the presence of a noisy benchmark, which is itself a byproduct of the characteristics of the underlying assets, leads to effort shirking by the portfolio manager, relative to the first best outcome. This result highlights a key limitation of linear contracts in the delegated portfolio management setting when perfect observability of asset returns is not feasible, as is often the case.

Are there contracting solutions that can help partially address the above shortcomings of the standard linear contracts? And if so, how similar are these alternative contracts to the kind of contracts we observe in practice in the case of say, hedge funds and private equity funds? We turn next to an examination of this question. Our analysis suggests two major results. One, the second-best optimal contract in this setting features an option-type component in addition to a component that is linear in performance. Two,

² For example, in the extreme case, an equity index fund is likely to have a correlation close to one with respect to say, the S&P 500 index.
the relative importance of the option-type component is an increasing function of the asset’s opacity. In particular, the weight of the option–type component in the optimal second-best contract increases with asset opacity. Further, the principal’s utility loss from restricting the weight of the option-type component to zero, is increasing in the asset’s opacity. These results provide an intuitive justification for the differences between the kinds of portfolio management contracts typically observed in the different segments of the asset management industry. In particular, the results provide a rationale for the option-like incentive contracts observed in the alternative asset management industry including hedge funds and private equity funds.

Our paper is related to an extensive literature that explores the principal-agent problem in the delegated portfolio management context. Following the pioneering work of Holmström and Milgrom (1987), a number of studies have demonstrated the optimality of linear contracts in the generic principal-agent setting. By contrast, Stoughton (1993) highlights an important shortcoming of linear contracts in the delegated portfolio management setting. Typically, in this setting, the portfolio manager first exerts costly effort to obtain a private information signal and subsequently determines the portfolio allocation. By changing the portfolio allocation the manager can alter the scale of the response to the signal thereby limiting the effectiveness of linear contracts in addressing the managerial effort underinvestment problem. Based on a similar intuition, Admati and Pfleiderer (1997) highlight the limitations of benchmark-linked linear contracts. Similarly, Starks (1987) finds that a symmetric contract does not eliminate the effort underinvestment problem in a setting where the appropriate risk sharing and effort incentives are both of concern. Stoughton (1993) and Bhattacharya and Pfleiderer (1985) explore quadratic contracts within a security-analyst context in which the portfolio manager simply reveals his information to the investor. However, as noted by Stoughton (1993), such contracts are not feasible in the more realistic delegated portfolio management setting.

In a recent paper Ou-Yang (2003) demonstrates the optimality of linear contracts in a particular form of the delegated management problem. On the other hand, Li and Tiwari (2009) adopt Stoughton’s (1993) framework to show that an appropriately designed option-type bonus fee contract can in fact be
used to improve efficiency and such a contract dominates all symmetric contracts. The present paper extends the literature along two dimensions. One, it helps clarify the conditions under which the first best outcome may be achieved via linear contracts in the delegated portfolio management setting, and by extension the conditions under which such contracts are sub-optimal. Two, it helps explain the observed differences in the nature of the portfolio management contracts across different kinds of investment vehicles. Importantly, the paper helps explain the existence of asymmetric option-type contracts in the alternative asset universe including hedge funds and private equity funds without appealing to market frictions such as high entry costs and imperfect competition in the industry.³,⁴

The rest of the paper is organized as follows. Section I explores the conditions under which the first best outcome may be achieved via linear contracts in the delegated portfolio management setting. It shows that a key condition is the assumption that the manager’s portfolio allocation is not based on private information, and that the investor can noiselessly observe the asset returns. Section II examines the contracting problem in the case where the manager’s portfolio is invested in relatively opaque assets. The section also presents results from a numerical analysis that examines how the form of the optimal contract changes and how the principal’s utility varies with a change in the underlying portfolio’s opacity. Concluding remarks are presented in Section III.

I. Contracting in Delegated Portfolio Management with Complete Asset Information

We begin by identifying some key features of the information environment in the delegated portfolio management problem that are essential for linear contracts to be applicable in such a setting. Our basic argument is built on a simple observation, which we illustrate in this section using a specific framework. The intuition is not restricted to the specific setup. More broadly, the underlying intuition in its general

³ In related work Das and Sundaram (2002) show that in a framework with differential managerial ability and imperfect competition in the market for managers/advisors, investor welfare is generally higher under a regime where only the option type “bonus” performance incentive fee is allowed in the contract relative to a regime where only the “fulcrum” fee is allowed. Studies that focus exclusively on the risk taking incentives include Carpenter (2000) and Grinblatt and Titman (1989) while Palomino and Prat (2003) explore a setting which abstracts from the risk sharing concern.

⁴ While not the focus of the present paper, a number of studies have explored the economics of the commonly observed high-water mark contracts in hedge funds. These include Goetzmann, Ingersoll, and Ross (2003), Panageas and Westerfield (2009), and Lan, Wang, and Yang (2012).
form allows us to compare several models, including Holmström and Milgrom (1987), Stoughton (1993), and Li and Tiwari (2009). It also allows us to uncover the essential source of the difference in the key results of Ou-Yang (2003) and Li and Tiwari (2009).

I.A. The Basic Model

In this section, we show that when the asset price (process) is perfectly observable, a linear contract can achieve the first best solution. This result serves as the benchmark for our main analysis in the next section where partial observability of the asset price is allowed. In the current section we also discuss how such a result is related to some recent results in the literature.

In the model, a principal (investor) contracts with an agent (manager) to manage her wealth. The agent has access to an investment opportunity set characterized by a risk-free asset with gross return $R_f$ realized at the end of the period, and a set of risky assets with gross return denoted by the random vector, $R$. The agent decides the allocations denoted by the vector, $A$, to each of the risky assets, with the rest of the wealth being invested in the risk-free asset.

Throughout, we will assume that the portfolio’s terminal value, which we denote by $W$, is observable by both parties and therefore can be contracted on. In general, we have the following result.

Proposition 1. Let the agent’s compensation scheme, denoted by $S^*$, take the following form:

$$S^* = W - W^*_p,$$  \hspace{1cm} (1)

where the term $W^*_p$ is the payoff to the principal under the first best solution. Assume that the portfolio performance ($W$) is fully determined by the agent’s portfolio allocation vector, $A$, the assets’ realized return $R$, and the risk free rate, $R_f$.\footnote{We suppress the time subscript for brevity. All of the items ($A$, $R$, and $R_f$) can in fact be time varying.} Assume that all the assets in the portfolio are publicly traded and therefore the investor can observe the asset return, $R$. Further, assume that the first best asset allocation vector, $A^*$, is based on public information. The first best outcome is achieved by the contract in (1), and the agent will voluntarily choose the first best portfolio allocation.
Proof. Under the assumptions of the proposition, the principal’s payoff in the first best scenario, $W_p^*$, depends on $A^*$, $R$, and $R_f$. In other words, $W_p^*$ is based only on public information given the assumption about the first best allocation, $A^*$. Therefore, the contract in (1) is feasible. It is clear that under the compensation scheme, $S^*$, described in Equation (1), the principal’s payoff, $W_p$, is given by, $W_p = W - S^* = W_p^*$, which is in fact the first best outcome. Therefore, the first best outcome is achieved.

The fact that the agent chooses the first best portfolio allocation can be shown accordingly.

The key assumptions underlying the above result are that (a) the manager’s portfolio allocation is not based on private information, and (b) the principal is able to observe the asset returns. These assumptions ensure that the principal can determine the appropriate first best payoff, $W_p^*$, to be demanded from the agent as part of the portfolio management contract, which in turn ensures that the first best portfolio allocation is realized. It is important to note that the conditions in the proposition do not rule out the possibility that the manager may have superior abilities compared to the investor. For instance, the manager may be able to implement trades at the best possible transaction price while minimizing price impact. Furthermore, the proposition’s assumptions still allow for the possibility that the manager’s actions are unobserved by the investor. In general, the investor cannot infer the manager’s portfolio allocation ex post. Further, the manager alone bears the private cost of his actions. Nevertheless, the moral hazard problem is completely resolved under the above assumptions. Also, note that the contract specified in (1) is linear in the final outcome, $W$. In particular, under the contract, the manager’s compensation is determined by comparing the portfolio’s performance, $W$, to a benchmark, $W_p^*$.

Admittedly, the above “proof” is heuristic in nature. In the technical appendix we include a detailed discussion that first properly sets up the problem and then derives a fuller version of Proposition 1 which applies more broadly to principal-agent problems beyond the delegated portfolio management setting. Note that the proposition clearly applies to the trivial case where the agent’s action can be verified ex post, because in this case, the compensation scheme in (1) is based on observables and therefore is
enforceable. As another familiar example where the proposition applies, assume that the agent’s action cannot be verified ex post, but the agent is risk neutral. Consider, for instance, the framework explicitly analyzed by Shavell (1979). It is shown by Shavell that in one of the first best solutions for this case, $W_p^* = k$, where $k$ is a constant. Therefore, the contract in Proposition 1 takes the form, $S^* = W - k$, which is feasible under the assumption that the performance $W$ is ex post verifiable and therefore can be contracted on. Therefore, the first best outcome for the principal is achieved by the contract, $S^*$.

To summarize, in this section, we have shown that linear contracts can be relied on to achieve the first best outcome in delegated portfolio management. In general, the key assumptions that are at the core of the argument are:

a) The first best portfolio strategy is based on public information.

b) The principal can observe the returns on the available assets.

Under the above assumptions, linear contracts can be optimal in the delegated portfolio management context.

I.B. Application to Recent Results in Delegated Portfolio Management

We next apply Proposition 1 to help shed light on some of the recent results in the literature on delegated portfolio management. First, we show that Ou-Yang’s (2003) main results can be quickly derived as corollaries to Proposition 1. We then examine the relation of the results in the previous section to other relevant frameworks in the literature, in particular, Holmström and Milgrom (1987), Stoughton (1993), Li and Tiwari (2009), and Edmans and Gabaix (2009).

1. A particular set of specifications for functional forms and distributions

In the setting examined by Ou-Yang (2003), all assumptions underlying Proposition 1 are satisfied. In particular, the first best portfolio allocation is based on public information and the principal can observe the asset returns. The fully specified model in Ou-Yang’s paper includes the following assumptions for the price processes, the wealth process, the cost function, and the utility functions. The
risk-free rate is constant and denoted by \( r \). The price-process of the risky assets is described by the following geometric Brownian motion:

\[
dP_t = \text{diag}(P_t)(\mu dt + \sigma dB(t)),
\]

where \( \mu \) is a constant vector in \( R^N \), \( \sigma \) is a constant matrix in \( R^{N \times d} \) with linearly independent rows, and \( B \) is a \( d \geq N \) dimensional standard Brownian motion. The portfolio manager chooses the dynamic allocations to the risky assets (and the risk-free asset) over time. The allocations are however, not observed by the investor. Let the dollar amount invested in the risky assets at time \( t \) be represented by, \( A_t \). The wealth process \( \{W_t\} \) for the portfolio strategy \( \{A_t\} \) is then given by

\[
dW_t = [rW_t + A_t^T h] dt + A_t^T \sigma dB_t,
\]

where \( h \equiv \mu - r \cdot 1 \), and \( 1 \) denotes the unit vector. The instantaneous cost function is specified as

\[
c(t, A_t, W_t) = \frac{1}{2} A_t^T k(t) A_t + \gamma W_t,
\]

where \( k(t) \) is an \( N \times N \) matrix, and \( \gamma \) is an constant. The agent’s preference over wealth is described by

\[
U_a(W) = -\frac{1}{R_a} e^{-R_a W}, \quad \text{and the principal’s utility is } U_p(W) = -\frac{1}{R_p} e^{-R_p W}.
\]

While Ou-Yang does not present the first best solution, we provide a solution in Appendix I. Briefly, we note that the manager’s time \( t \) certainty equivalent utility along the optimal path is a random process. As a matter of formality, we can use this process to substitute out the manager’s participation constraint. This substitution leads to a restatement of the original dynamic programming problem, which then allows for a textbook standard solution. A complete presentation of solution and the steps that lead to it are provided in Appendix I. We summarize the relevant component of the result in the following proposition.

**Proposition 2.** The first best solution of the optimal portfolio policy is given by

\[
A_t^* = f(t) \left[ k_t + \frac{R_a R_p}{R_a + R_p} f^2(t) \sigma \sigma^T \right]^{-1} h,
\]
where \( f(t) = (1 - \frac{\gamma}{r})e^{r(t-t')} + \frac{\gamma}{r} \). The investor’s payoff is

\[
W_p^* = -F + \frac{R_a}{R_a + R_p} \int_0^T f(t) A_t^T \text{diag}(P_t)^{-1} dP_t
\]

(6)

where \( F \) is a constant.

The optimal contract follows as a corollary to Proposition 1.\(^6\) Combining Equation (1) and Equation (6), an optimal contract in this setting can be expressed as

\[
S_T = W_T - W_p^* = F + \frac{R_p}{R_a + R_p} W_T + \frac{R_a}{R_a + R_p} \left[ \int_0^T f(t) A_t^T \text{diag}(P_t)^{-1} dP_t \right],
\]

(7)

which is exactly the same as Equation (10) in Theorem 1 in Ou-Yang (2003, page 185).

As Ou-Yang notes, the contract in Equation (7) may be viewed as consisting of three parts: (a) a fixed fee, (b) a proportional fee, and (c) a bonus or penalty that depends on the portfolio performance relative to an active benchmark. The bonus fee is symmetric in form – a result that appears to provide theoretical support for the current regulation that restricts the performance-based fees for investment company advisers to be of the symmetric form. Ou-Yang further develops some other forms of the optimal contract in the case where the cost function \( c \) is constant. These results are contained in his Theorem 2 (p. 188).

For these cases, consider the following compensation schedule for the agent:

\[
S = S^* + \lambda \left[ (A - A^*)(R - R_f) \right],
\]

(8)

where \( \lambda \) is a non-zero constant. It can be readily shown that all contracts in the form of (8) are optimal.

The derivation of the above result is provided in Appendix I (See Proposition A.2 in Appendix I).

By restricting the value of the parameter, \( \lambda \) in Equation (8) to the interval \((0,1)\), we arrive at Theorem 2 in Ou-Yang. It is interesting to note, however, that there is no need to restrict the parameter \( \lambda \) in an interval as Ou-Yang did, as long as \( \lambda \) is not zero.

\(^6\) While Proposition 1 in Section I is derived under some specific assumptions that are not directly applicable to the current situation, the same result can also be derived in a completely general setting that indeed embeds Ou-Yang’s setup as a specific case. The general derivation is provided in Appendix I.
I.C. Other Frameworks

From the above discussion, it is evident that the optimal contract derived by Ou-Yang (2003) is in principle identical to the contract we derive in Proposition 1 and its feasibility hinges on the assumptions listed at the end of Section IA. Importantly, the moral hazard problem is completely resolved under these assumptions. As a key point of departure, in the framework adopted by Stoughton (1993) and Li and Tiwari (2009), the manager expends effort to collect private information that is correlated with the asset returns. Consequently, linear contracts are no longer optimal in the framework of these two studies.

We use Stoughton’s framework to illustrate the point. The manager has access to a risk-free asset with gross return \( R_f \) and a risky asset with gross return, \( \mu + R_e \), where \( \mu \) is the mean and thus \( E(R_e) = 0 \). One of the key assumptions made by Stoughton is that the manager can observe a signal that is correlated with the true return:

\[
I = R_e + \varepsilon,  \tag{9}
\]

where all variables are assumed to be jointly normal. Furthermore, \( E(\varepsilon) = 0 \), and \( E(R\varepsilon) = 0 \). The information precision is captured by, \( \rho = \sigma_R^2 / \sigma_\varepsilon^2 \), which is also identified as the manager’s effort. The utility functions of the manager and the investor are assumed to be exponential:

\[
U_a(W_A) = -\exp\{-aW_A + V(\rho)\},  \tag{10}
\]

\[
U_p(W_B) = -\exp\{-bW_B\},  \tag{11}
\]

where \( W_A \) and \( W_B \) are the end-of-period wealth of the agent and the principal, respectively, and \( V(\rho) \) is the disutility of effort. At the first best outcome, the payoff to the principal is the following function of \( R \) and \( I \):

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7 We adopt notation that differs from Stoughton (1993) in order to be consistent with the other part of this paper.
\[ W_p^* = \frac{1/b}{(1/a) + (1/b)} W^* - \Phi \]

\[ = \frac{1/b}{(1/a) + (1/b)} \left[ W_0 R_f + \frac{a + b}{ab} \frac{\rho^* + 1}{\sigma_R^2} \left( \mu + \frac{\rho^*}{\rho^* + 1} (1 - R_f) (R - R_f) \right) \right] - \Phi, \]

where \( \Phi \) is a constant such that the manager’s participation constraint is satisfied. All the parameters and variables in the last row of the above expression are either known or observable at the end of the game to the investor and therefore, can in principle be contracted on, with the exception of the manager’s signal, \( I \). Due to this fact, the contract in Equation (1) is not feasible in Stoughton’s (1993) framework. Based on a similar setting as Stoughton, Li and Tiwari (2009) show that all forms of symmetric contracts are suboptimal. They demonstrate that an appropriately designed option-type bonus fee contract can be used to improve efficiency and such a contract dominates all symmetric contracts in this setting.\(^8\)

It is important to note that the insight of Holmström and Milgrom (1987) is based on a framework characterized by asymmetric information. In the continuous-time game analyzed by them, the agent incurs a cost and controls the mean (\( \mu \)) of the output diffusion process (\( Z_t \)):

\[ dZ_t = \mu_t dt + dB_t, \quad (13) \]

where \( B_t \) is a driftless \( N \)-dimensional vector standard Brownian motion. An important condition that leads to the optimality of a linear contract is the assumption that the principal gets to observe only the total output, \( 1' Z_t \), where \( 1 = (1,1,...,1)' \). This is in contrast to the requirement of Proposition 1, where a term in the contract specified by Equation (1) is a contingent payoff, \( W_p^* \), which does not depend on the agent’s actions. The only feasible contingent payoff in the Holmström and Milgrom (1987) framework that does not depend on the agent’s actions is a constant payoff. This is due to the fact that the principal in this framework does not observe anything beyond the total output. Clearly, such a constant payoff cannot

\(^8\) It is worthwhile to highlight an important distinction between the framework adopted in the studies by Stoughton (1993) and Li and Tiwari (2009) and that in Holmström and Milgrom (1987). Both of the former studies base their analysis on a stationary model in contrast to Holmström and Milgrom (1987), who demonstrate the optimality of a linear contract in a dynamic setting.
be equal to $W_r^*$ in the case in which the agent is risk averse. This sufficiently demonstrates that the contract in (1) does not apply to the Holmström and Milgrom setting.

In a recent study Edmans and Gabaix (2009) show that the Holmström and Milgrom result on the optimality of linear contracts can be achieved in settings that do not rely on assumptions such as exponential utility, continuous time framework, and Gaussian noise. It is natural to ask whether this generalization extends to the delegated portfolio management setting. A key aspect of the Edmans and Gabaix generalization is that it relies on a framework in which information is revealed to the agent before the agent’s action is chosen. Furthermore, while the model has a multiple period structure, the potential complexity from such a structure is minimized. This is achieved through two key assumptions. First, the cost function of the agent’s action is assumed to take a pecuniary form. Therefore, the aggregation of the overall cost to the agent of his actions in multiple periods can be achieved by adding up the pecuniary cost incurred in each individual period. The second assumption is that at the end of each period, the realization of the outcome, jointly determined by nature and the agent’s action, is publicly observed. By these assumptions, each period’s game is sufficiently independent, and the aggregation through multiple periods can be achieved rather mechanically. In stark contrast, as emphasized by Stoughton (1993), the interesting feedback effect of actions in different stage of the game in delegated portfolio management is what makes the contracting problem particularly challenging. More specifically, the agent in such an environment undertakes costly effort prior to the realization of a noisy signal related to future asset payoffs, and he is then required to decide on the asset allocation under imperfect information. Hence, the limitations of the Homstrom-Milgrom framework in the context of delegated portfolio management, first highlighted by Stoughton (1993), are still valid.

II. The Case of Opaque Assets

As demonstrated in the previous section, our basic result, that the first best solution can be implemented in the delegated portfolio management setting, depends critically on the sufficiency of the available information regarding the returns of the traded asset(s). The implementation involves a
benchmark that the manager is measured against. In particular, the benchmark tracks the performance of the first best portfolio strategy. It leads naturally to the question: what if the investor can only construct a noisy benchmark, a benchmark that tracks the first best strategy only imperfectly? In this section we seek answers for this question.

We build on the basic setup in Section I. For simplicity, we reduce the number of risky assets traded by the manager to 1. Further, we assume that the risky asset is ‘opaque.’ Specifically, we assume that instead of observing the return on the risky asset in the portfolio, the investor can only observe the return on a benchmark reference asset that has the same marginal return distribution, but is only imperfectly correlated with the risky asset.

The model analyzed here is a stationary one. As before, a principal contracts with an agent to manage her wealth. We normalize the initial investment to $1. With this assumption, the return and the terminal value of the investment are identical. We assume that the rate of return on the risky asset follows the geometric Brownian motion, so that the asset payoff is never negative. At the beginning of the period, the manager decides the portfolio weights, $A$, which are held constant throughout the period. With the above assumption, the risky asset return has a log-normal distribution. The log rate of return for the portfolio is:

$$w = A(r - r_f) + r_f,$$  \hspace{1cm} (14)

where $r$ is the log return of the risky asset and $r_f$ is the log risk free rate of return. We denote the terminal value of the portfolio by $W$, and we have $W = e^w$. We assume that there is a benchmark asset, whose return is denoted by $r_b$, and this benchmark asset helps the investor track the performance of the risky asset. Specifically, we assume that $r$ and $r_b$, are jointly normally distributed with the identical marginal distribution, $N(\mu, \sigma^2)$. We denote the correlation between $r$ and $r_b$ by $\rho$, where $0 < \rho \leq 1$. Given this distributional assumption, the rate of return on the risky asset may be expressed as:

$$r = (1-\rho)\mu + \rho r_b + \sqrt{1-\rho^2}\sigma \varepsilon,$$  \hspace{1cm} where the noise term, $\varepsilon$, is independent of $r_b$ and has a standard Normal distribution.
The principal is assumed to be risk neutral while the agent is risk averse with the log utility function: \( U_a(C) = \log(C) \), where \( C \) denotes the agent’s compensation. The agent suffers a disutility, \( V(A) \), of managing the portfolio. Overall, the agent’s utility is given by \( U_a(C) - V(A) \), where the agent’s compensation, \( C \), is subject to the restriction \( C \geq c \) to account for the limited liability feature. The constant \( c \) is set to be a small positive number.\(^9\) The agent’s reservation utility is assumed to be, \( U_r(p) \), where the constant \( p \) is to be interpreted as the agent’s opportunity cost of entering the contract with the principal.

Note that when \( \rho = 1 \), the principal can noiselessly observe the traded risky asset’s return, thereby satisfying the key requirement of Proposition 1. It follows that the first best solution can be achieved in the case when the correlation between the risky asset and the benchmark asset equals 1. The first best portfolio allocation is determined by the following maximization problem:

\[
A_{FB} = \arg \max_A R_f \exp \left( A(\mu - r_f) + \frac{1}{2} A^2 \sigma^2 \right) - pe^{V(A)},
\]

(15)

We assume that the agent’s cost function is of the form, \( V(A) = \frac{1}{2} kA^2 \). We note that such a function has the following properties: \( V(0) = V'(0) = 0 \), and with a sufficiently large choice of \( k \), the objective function in (15) is strictly concave.

The question we focus on in this section is: what happens when the principal’s information is less than perfect, i.e., when \( \rho < 1 \). We begin by noting that in this case the benchmark will not be perfect and the first best solution will not be achieved. Below we examine the form of the second best optimal contract in this case. The contract is dependent on the observables. In particular, we write the contract as \( C(r_b, w) \), a function of the log return on the benchmark asset, \( r_b \), and the log portfolio value, \( w \). A key to the analysis is the question of how the probability distribution of the log portfolio return, \( w \), depends on

\(^9\) To avoid the singularity of the log function at zero, we bound the compensation away from zero.
the portfolio allocation, \( A \), and the observation of the benchmark log return, \( r_b \). Note that, conditional on observing \( r_b \), the distribution of the risky asset’s return, \( r \), is \( N((1-\rho)\mu + \rho r_b, 1-\rho^2) \). We denote \( \mu_b = (1-\rho)\mu + \rho r_b \). The conditional distribution of the log return of the portfolio, \( w \), is given by \( N(r_f + A(\mu_b - r_f), A^2(1-\rho^2)\sigma^2) \). The log p.d.f. of the terminal portfolio value, \( W \), is given by:

\[
\log f(W, r_b; A) = -\frac{1}{2} \log (2\pi(1-\rho^2)\sigma^2) - w - \log(A) - \frac{(r_f - A(\mu_b - r_f))^2}{2A^2(1-\rho^2)\sigma^2}.
\]  

(16)

By the likelihood ratio principle (see Holmström (1979)), the solution of the optimal contract problem is characterized by the following set of equations that jointly solve for the compensation, \( C \), portfolio allocation, \( A \), and parameters, \( \lambda \), and \( \varphi \). First, the optimal contract should set the compensation \( C \) to satisfy the following equation:

\[
\frac{U_b'(W-C)}{U_a'(C,A)} = \lambda + \varphi \cdot \frac{\partial}{\partial A} \left( \log f(W, r_b; A) \right).
\]  

(17)

Then, the shadow price of the incentive compatibility constraint, \( \varphi \) is the solution to the adjoint equation,

\[
0 = E\left[ (e^w - C(r_b, w; A)) \frac{\partial \log f(W, r_b; A)}{\partial A} \right] + \varphi E\left[ \left( \frac{\partial \log f(W, r_b; A)}{\partial A} \right)^2 + \frac{\partial^2 \log f(W, r_b; A)}{\partial^2 A} \log(C(r_b, w, A)) \right] - V''(A),
\]  

(18)

and the value of the parameter \( \lambda \) is the solution to the participation constraint:

\[
E\left[ \log(C(r_b, w; A)) \right] - V(A) = \log(p).
\]  

(19)

Finally, the portfolio allocation in the second best solution is given by the first order condition of the agent’s problem:

\[
E[h_1 \log(C(r_b, w; A))] - V'(A) = 0.
\]  

(20)

Note that the contract compensation, \( C \), depends only on \( r_b \), and \( w \), and not directly on the manager’s portfolio allocation, which is of course unobservable to the investor and therefore cannot be
directly contracted on. However, through Equation (17), the portfolio allocation, $A$, in the second best solution, does influence $C$ as a parameter. Therefore, we use the notation $C(w, r_b; A)$ to track this dependence in the above equation system. After the portfolio allocation $A$ is determined, its value is plugged in to solve for $C$. From Equation (17), we have that the optimal contract takes the form:

$$C(w, r_b) = c + \left[ \lambda + \phi \cdot \frac{\partial}{\partial A} \left( \log f(W, r_b; A) - c \right) \right]^+$$

$$= c + \left\{ \lambda + \phi \left[ -1 + \frac{(w-r_f - A(\mu_b - r_f))(\mu_b - r_f)}{A^2(1-\rho^2)\sigma^2} + \frac{(w-r_f - A(\mu_b - r_f))^2}{A^3(1-\rho^2)\sigma^2} \right] - c \right\}^+$$  \hspace{1cm} (21)

A notable feature of the above contract is that the manager’s compensation is based on a comparison of the portfolio’s performance, $w$, against the benchmark: $r_f + A(\mu_b - r_f)$. Here, $A$ is the portfolio allocation in the second best solution, and it does not vary with the manager’s off-equilibrium portfolio allocation. Note also that the above second best contract is convex.

**Proposition 3.** When the principal’s information is not perfect (i.e., $\rho < 1$), the shadow price of the incentive compatibility constraint is positive. That is, $\phi > 0$. The portfolio allocation is lower than the first best allocation, i.e., $A^* < A_{FB}$.

**Proof.** See Appendix II.

The above proposition shows that due to imperfect information (i.e., when $\rho$ is strictly less than 1), the manager’s allocation to the actively managed portfolio will be less than that in the first best solution, which is a form of effort shirking in our setting. Hence, the first best outcome is achieved only in the special case with perfect observability of the risky asset’s return, i.e., when $\rho = 1$.

**II.A. Convergence to First Best**

We next examine how the second best outcomes converge to the first best as the correlation between the risky asset and the reference benchmark asset approaches 1. From the results in Section I, we know that in the special case when $\rho = 1$, the first best outcome is achieved. The contract in Equation (1) takes
the following form in this case: \( C_{FB}(w, r_b) = W - R_f \exp\left[ A_{FB}(\mu_b - r_f) \right] + \lambda \), where \( \mu_b = (1 - \rho)\mu + \rho r_b \) and \( \lambda \) is chosen so that the agent’s participation constraint is binding at the equilibrium. Given that \( \rho = 1 \), we have, \( \mu_b = r_b = r \). As shown in Section 1, this contract will lead to the first best allocation, and at the equilibrium, the contract will pay the agent a constant amount which is consistent with optimal risk sharing in this setting. Further, we can restrict the agent’s payment to be nonnegative in the above contract as such a constraint will not be binding at the equilibrium.

We now turn to the case where \( \rho \neq 1 \). To be comparable with the second best contracts discussed earlier, we constrain the contract \( C_{FB}(w, r_b) \) away from zero. That is, we consider the contract \( C'_{FB}(w, r_b) = c + \left[ W - R_f \exp\left( A_{FB}(\mu_b - r_f) \right) + \lambda - c \right] \). Note that, given this contract, the benefit to the manager from deviating from the first best solution, is bounded by a function of the deviation of his portfolio allocation from the first best allocation, \( A - A_{FB} \). The cost to the manager of deviating from the first best is the possibility of getting the minimum payment, \( c \). For any \( \varepsilon > 0 \) and \( | A - A_{FB} | > \varepsilon \), we have the cost increasing to infinity as \( c \) approaches zero uniformly. Therefore, as \( c \to 0 \), the agent’s portfolio allocation choice when facing the above contract also approaches the first best. In other words, when we choose \( c \) to be sufficiently small, the agent’s portfolio choice is practically the same as the first best solution. Given that the second best solutions for the cases with \( \rho < 1 \) should weakly dominate the outcome from the contract, \( C'_{FB} \), the second best solutions will converge to the first best solution as \( \rho \) approaches 1. We summarize the conclusion in the following proposition.

**Proposition 4:** As \( \rho \to 1 \), and \( c \to 0 \), the second best solution approaches the first best.

### II.B. Numerical Analysis

How do the parameters in the optimal contract, the manager’s portfolio allocation, and the principal’s utility change with a change in the correlation between the risky asset and the benchmark asset, \( \rho \)? In particular, how do the contract parameters, the portfolio allocation and the principal’s utility change as
\( \rho \) declines substantially below 1, i.e., as the manager’s portfolio progressively becomes more opaque? We rely on numerical analyses to address these questions. To calibrate the model, we assume that the asset return \( R \) in the model has the same statistical characteristics as the broad market index. Using the monthly U.S. T-bill and value-weighted market index returns for the period 1963:01-2011:09, we get the following annualized statistics: average risk free rate of 5.19\%, average market excess return equal to 5.186\%, and market volatility of 15.6\%.

We further assume that the agent’s reservation utility is equal to 2\% of the initial assets under management. The cost function parameter, \( k \), is set equal to 1.

For the purpose of comparison, we study the outcomes under the following three contracts as the correlation coefficient, \( \rho \), varies between 0 and 1: (a) the second best optimal contract, (b) the optimal linear contract, and (c) the practical incentive contract with an option-like bonus fee. The last contract is a contract that is similar to the kind of incentive contract observed in practice. The second best optimal contract is as described in (21). To facilitate comparison, we now rewrite this contract in the following form:

\[
C(w, r_b) = F + \beta \left( w - r_f - A(\mu_b - r_f) \right) (\mu_b - r_f) + \gamma \left( w - r_f - A(\mu_b - r_f) \right)^2
\]

We can interpret the first term \( F \) in the above contract as the fixed salary component. The second component can be interpreted as a linear component, and the third component as a convex component. We note that the parameters \( \beta \) and \( \gamma \) in the above expression are related by: \( \beta / \gamma = A \). Furthermore, the linear component is linear in the portfolio’s performance. However, the component is in the form of the interaction between the portfolio performance and the performance of the benchmark. Therefore, the pay-performance sensitivity in this case varies with the benchmark performance. We impose the constraint, as in our analytical study, that the manager’s compensation is no less that a predetermined small number, \( c \), which is set equal to 0.0001. Such a constraint is imposed in all three contracts.

---

\( ^{10} \) As a robustness check we also calibrated the model using the market statistics for the following sub-periods: (a) 1926:07 – 2012:03; (b) 1995:01 – 2012:03; (c) 2000:01 – 2012:03; (d) 1963:01 – 1987:12; and (e) 1988:01 – 2012:03. In each case the results are qualitatively similar to the results for our base case presented here.
The linear contract we consider is a contract that relates the manager’s compensation to the portfolio performance in a linear fashion. That is,

\[ C(w, r_b) = F + \varphi(w - w_b), \]  

(23)

where \( F \) is the fixed salary, \( \varphi \) is the pay-performance sensitivity, and \( w_b = A_b (r_b - r_f) + r_f \). We set \( A_b \) at the equilibrium allocation, \( A \). Finally, for the practical incentive contract, we take the above linear contract as the starting point and add a component that resembles the usual option-type bonus fee:

\[ C(w, r_b) = F + \varphi(w - w_b) + \gamma(w - w_b)^+. \]  

(24)

The results of the numerical analysis are reported in Table 1. In each panel of the table, the first column lists the value of the correlation between the benchmark and the risky asset return, while the second column reports the principal’s utility in excess of the initial investment and as a percentage of the initial investment. The third column shows the corresponding asset allocation to the actively managed asset induced by the contract. In the following two (Panel B) or three (Panels A and C) columns, the contract parameters \((F, \beta, \gamma)\) are listed under the title “salary”, “linear”, and “quadratic” or “option”. The last two columns of Panel A report the relative weights of the two contract components: linear versus quadratic. The relative weights are based on the relative variation in the linear and quadratic component of the manager’s compensation as the performance of the actively managed asset and the benchmark asset varies across the joint distribution of the risky asset return and the benchmark return. For easy comparison, the last row of each panel lists the first best outcomes in terms of the portfolio allocation and the principal’s utility.

From Panel A, it is clearly seen that under the second best contract, when the correlation coefficient, \( \rho \), is less than one, there is an underinvestment in the actively managed risky asset. In the first best outcome, the allocation to the risky asset is 146.77% with the resulting principal’s utility at 8.75%. The induced allocation in the second best case when \( \rho = 0 \) is 89.08%. The principal’s utility in this case drops to 6.51%, i.e., the principal suffers a utility loss of more than twenty five percent as the benchmark
asset’s correlation drops from a perfect 1 to zero. The underinvestment problem is less severe as the correlation, $\rho$ increases. Indeed, as $\rho$ increases, both the allocation to the risky asset and the resulting principal’s utility increase gradually. Furthermore, as $\rho$ approaches 1, both the risky asset allocation and the principal’s utility converge to their corresponding values in the first best outcome.

The second best contract needs to achieve the twin objectives of providing the appropriate effort incentive to the manager and achieving the appropriate risk sharing between the two parties. As $\rho$ decreases, the performance of the benchmark reference asset is less informative about the manager’s actions, i.e., his risky asset allocation. As a result, as $\rho$ decreases, i.e., as the portfolio becomes increasingly opaque, the linear component of the contract becomes less effective, and the contract relies more on the convex component to motivate the manager to invest in the risky asset. Indeed, as seen from Panel A of the table, the relative weight of the convex component increases substantially as $\rho$ decreases.

When comparing the linear contract outcomes (Panel B) with the second best outcomes (Panel A), we can see that there is substantial utility loss for the principal when $\rho$ is low. For instance, when $\rho = 0$, the utility decreases from 6.51% in the second best case to 3.86% in the linear contract case – a utility loss of over 40%. The loss of utility in the case of linear contracts is largely due to the underinvestment in the risky asset. For instance, when $\rho = 0$, the allocation to the risky asset is 89.08% in the case of the second best solution, while it is only 12.65% in the case of the linear contract. As $\rho$ increases, the allocation to the risky asset in the linear contract case increases substantially with the resulting decline in the principal’s utility loss. For example, when $\rho = 0.98$, the utility loss under the linear contract is less than 1% relative to the second best case.\(^{11}\) This result highlights the fact that as the underlying portfolio becomes less opaque the linear contract becomes more attractive. By Proposition 4, the principal’s utility converges to first best in both cases when $c$ approaches zero.

\(^{11}\) Some residual difference remains due to the constraint that all contract payments have to be larger than a positive constant, $c$. 
We next consider the outcomes under the practical incentive contract (Panel C) which features the option-like component in addition to the linear contract component. We note that the underinvestment problem in the linear contract is alleviated to a certain degree by the inclusion of the option-type component in the practical incentive contract. In the case when \( \rho = 0 \), the portfolio allocation to the risky asset increases from 12.65% in the linear contract case to 27.04% in the case of the contract with the option-like component. The corresponding investor’s utility increases from 3.86% in the linear contract case to 4.33% in the case of the practical incentive contract. The gain in the investor’s utility is much less for higher values of \( \rho \). Indeed, when \( \rho \) gets close to 1, the option-like component is no longer useful. In fact, as seen from Panel C of Table I, when \( \rho = 0.90 \), the option-like component coefficient is close to zero, and the coefficient becomes zero when \( \rho = 0.95 \) or when \( \rho = 0.98 \). This is an interesting result as it confirms that as the ‘opacity’ of the manager’s portfolio declines, the option-like component is no longer needed to motivate the manager, and the linear contract component suffices. Conversely, it is precisely in the case of non-traditional, relatively ‘opaque’ assets that the option-type component is a desirable contract feature.

**Concluding Remarks**

This paper addresses two related issues in the context of delegated portfolio management. First, it explores the conditions under which first best outcomes may be achieved via linear contracts in this setting. The analysis shows that a benchmark-linked linear contract can achieve the first best outcome when (a) the manager’s portfolio allocation is not based on private information, and (b) the principal is able to noiselessly observe the asset returns. Intuitively, with perfect observability of asset returns, the investor can determine the first best portfolio outcome and penalize any deviation from this benchmark value.

Second, the paper explores the limitations of the linear contract when the portfolio is invested in ‘opaque’ assets that are illiquid or that are privately traded. This is a case of particular interest as it characterizes the portfolios of a number of alternative or non-traditional investment vehicles such as
hedge funds and private equity funds. The relative opacity of the portfolios in these cases contributes to the benchmarks for such assets being relatively noisy. We examine how the optimal contract changes as the correlation between the manager’s portfolio and the reference benchmark portfolio declines. The analysis suggests that in the absence of perfect observability, i.e., when the correlation between the manager’s portfolio and the benchmark portfolio is less than 1, the first best outcome is no longer feasible. The second-best optimal contract in this setting features an option-type component in addition to a component that is linear in performance. Moreover, the relative importance of the option-type component is an increasing function of the portfolio’s opacity. Further, the principal’s utility loss from restricting the weight of the option-type component to zero is increasing in the asset’s opacity. These results help explain the differences in the contracts observed in the traditional asset management sector and the alternative asset management arena. In particular, they provide a rationale for the option-like incentive contracts observed in the alternative asset management industry.
References


Table 1

The table documents the numerical results on the outcomes of using three types of contracts under different scenarios where the correlation, \( \rho \), between the return on benchmark observed by the principal, and the return on the portfolio managed by the agent, varies from 0 to 0.98. The table reports, for each contract, the principal’s expected utility net of the initial investment (second column) expressed as a percentage of the initial investment, the agent’s allocation weight on the actively managed risky portfolio (third column), and the contract components. Panels A, B and C correspond to the second best contract, the linear contract, and the practical incentive contract, respectively. The contracts are described in more detail in Section II.B. in the text. Columns 4, 5 and/or 6, under the subtitle “Contract”, report the coefficients in the contract: coefficient \( F \) under “Salary”, coefficient \( \beta \) under “Linear”, and/or coefficient \( \gamma \) under “Quadratic” in Panel A and under “Option” in Panel C. To facilitate interpretation, in Panel A, we also report the relative weights of the two variable contract components. The relative weights are based on the relative (absolute) variation in the linear and quadratic component of the manager’s compensation as the performance of the actively managed risky asset and the benchmark asset varies across the joint distribution of the risky asset return and the benchmark return. The relative weights are reported in Columns 7 and 8 under the subtitle “Relative Weights”. The key inputs for the calibration are based on the following annualized U.S. market statistics for the period 1963:01-2011:09: average risk free rate of 5.19%, average market excess return equal to 5.186%, and market volatility equal to 15.6%.

Panel A. Second Best Contract

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<th>( \rho )</th>
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<th>Allocation</th>
<th>Salary</th>
<th>Linear</th>
<th>Quadratic</th>
<th>Relative Weights</th>
<th>Linear</th>
<th>Quadratic</th>
</tr>
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<td>89.08%</td>
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<td>0.293</td>
<td>17%</td>
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</tr>
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<td>90.42%</td>
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<td>0.278</td>
<td>0.307</td>
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<td>80%</td>
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<td>0.336</td>
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<td>77%</td>
<td>23%</td>
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<td>146.77%</td>
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### Panel B. Linear Contract

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<th>Linear</th>
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### Panel C. Practical Incentive Contract

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Appendix I (Technical Appendix)

Derivation of the results of Section I under a general framework

Below, we derive the results corresponding to those of Section I under a general framework. As will be demonstrated, our results do not rely on (a) the specific form of a utility function for the principal and the agent, (b) assumptions regarding how the agent’s action affects the distribution of a project’s payoff, and therefore the distributional assumptions regarding the state variables, and (c) the choice of a dynamic framework – namely, whether it is a continuous-time or a discrete-time model.

Consider an economy in which a principal hires an agent to carry out a business endeavor. The outcome of the endeavor is a wealth distribution along state space and time. We assume that the pair of a measure space and a sigma algebra, denoted by \((\Omega, F)\), captures all the dimensions and all the relevant distinctions in the state space and time within the model. To capture the idea that the agent’s action contributes to the outcome, we denote it in functional form as \(W(\cdot): A \rightarrow D\), where \(A\) is the space of all feasible actions by the agent, and \(D\) is the space of all possible wealth distributions along state space and time. That is, \(D\) consists of real valued \(F\)-measurable functions with domain on \(\Omega\), and a feasible action \(A \in A\) by the agent leads to a wealth distribution \(W(A) \in D\). We assume that the principal has a utility function, \(U_p : D \rightarrow R\). Such a general utility function ranks all feasible wealth distributions the principal potentially gets out of the joint venture.

The agent has a utility function over the compensation he receives and the action he chooses. Formally, \(U_A : D \times A \rightarrow R\). In the principal-agent setting, the agent’s action has to be voluntarily chosen. Therefore, an appropriately designed contract is needed to elicit the proper action from the agent. We view the contract from the agent’s perspective. As the outcome of any action \(A \in A\) the agent chooses, there are the resulting distributions (on the measure space \((\Omega, F)\)) of the compensation, which we denote by \(S(A) \in D\). That is, similar to \(W(\cdot)\), \(S(\cdot)\) is a function from \(A\) to \(D\). We denote the space of all feasible compensation-based contracts by \(S\).\(^{12}\) The generic contracting problem can be stated as the following optimization problem:

\[
\sup_{S \in S, A' \in A^*(S)} U_p \left( W(A') - S(A') \right), \tag{A.1}
\]

\[
s.t. \ A^*(S) = \arg \max_{A' \in A} U_a(S(A), A') \tag{A.2}
\]

\[
U_a(S(A^*), A^*) \geq \bar{U}_a. \tag{A.3}
\]

\(^{12}\) For a contract to provide motivation, it has to link the compensation with the agent’s action. Given that the agent’s action is not observable, this link is in general, not direct. For instance, \(S\) may be determined by \(A\) indirectly through \(W(A)\). Therefore, depending on the ex post information the principal has, the feasible contract space \(S\) is in general a proper subset in the space of all functionals from \(A\) to \(D\).
In the above, $\overline{U}_a$ is the agent’s reservation utility, and Inequality (A.3) is the agent’s participation constraint. Here, we assume that the principal takes the entire surplus and that the agent’s utility is driven to the reservation level. The additional constraint (A.2) is the agent’s incentive compatibility constraint. The notion of arg max in (A.2) is to be understood as the set of all solutions to the maximization problem. Therefore, $A^e$ is a set-value function such that for any $S \in S$, $A^e(S) \subset A$. As a special case, if $A^e(S)$ is always reduced to a single point for all candidate contracts, $S$, then $A^e$ is uniquely determined by $S$ through constraint (A.2). In this case, the only control variable for the optimization problem will be $S$, the choice of the contract. In the way we state the problem in (A.1), we adopt the convention that among all actions for which the agent is indifferent, he will choose the one that is most beneficial to the principal. Such a convention becomes unnecessary if we adopt the assumption that the agent keeps the entire surplus and the principal’s utility is driven to reservation level.

We assume that a first best solution exists. That is, the following optimization problem has at least one solution.

$$\max_{A \in A, S \in D} U_p(W(A) - S),$$

(A.4)

$$s.t. U_a(S, A) \geq \overline{U}_a.$$

(A.5)

For easy reference, we denote the set of first best solutions by $Q_{FB}$:

$$Q_{FB} = \{(A^*, S^*) | A^* \in A, S^* \in D, (A^*, S^*) \text{solves the optimization problem in (A.1) and (A.2)}\}$$

(A.6)

For the following discussion, we will in general use the notations $A^*$ and $S^*$ for the resulting quantities in the first best solution. The first best solution is typically understood as the optimal solution that can be achieved when the agent’s action can be ex post verified and therefore contracted on. Although this case is well understood in the literature, we make some remarks in order to facilitate our later comparison. Note that with a slight abuse of notation, $S$ in (A.4) and (A.5) is a point in the space in $D$, and it determines only one compensation distribution for the agent. In contrast, a contract $S(A)$ in (A.1)-(A.3) is a function from $A$ to $D$, and different actions by the agent can potentially lead to different compensation distributions for him. However, this inconsistency can be easily resolved following the general protocol of viewing a constant number as a constant function, by viewing the compensation distribution $S$ (therefore, a point in $D$) as equivalent to the constant function that maps any action $A \in A$ to the same compensation distribution $S$, with the understanding that there is an implicit dimension of the contract that threatens to punish the manager when his action deviates from the optimal action, $A^*$. This dimension of the contract is not explicitly stated in the problem, but it is viewed as trivial in intuition.

Comparing the optimization problems in (A.4)-(A.5) and (A.1)-(A.3), we note that the difference stems from the additional constraint, namely, the incentive compatibility constraint (A.2), for the contracting problem in (A.1)-(A.3). This constraint is the key factor that can lead to a potential moral
hazard problem. The potential moral hazard leads to a loss in efficiency if and only if constraint (A.2) is binding. For our purpose, a particular set of contracts are of interest. They take the form:

\[ S^{**}(A) = W(A) - (W(A^*) - S^*) \]  

(A.7)

for some pair \((A^*, S^*) \in Q_{FB}\). We denote the set of contracts taking the form in (A.7) by \(S^{**}\). That is,

\[ S^{**} = \{ S(\cdot) | S(\cdot) \in S, \text{ with } S(A) = W(A) - (W(A^*) - S^*) \text{ for some pair } (A^*, S^*) \in Q_{FB} \}. \]  

(A.8)

It will become clear later that this set of contracts can be intuitively understood as being equivalent to the principal “selling” the project to the agent. The key question about set \(S^{**}\) is whether it is empty. In case it is not empty, we have the following result:

**Proposition A.1.** If there exists a contract, \(S^{**} \in S^{**}\), then the incentive compatibility constraint in (A.2) is not binding and the first best solution is achieved by this contract. In fact, for any \(A^*\) in the set of \(A^*(S^{**})\), the pair \((A^*, S^{**}(A^*))\) is a first best solution. If we assume, in addition, that both \(U_p(\cdot)\) and \(U_a(\cdot)\) are continuous and strictly increasing in wealth, we have that \(A^* \in A^*(S^{**})\). That is, \(A^*\) is an optimal response to contract \(S^{**}\) for the agent.

**Proof:** Given the definition of \(S^{**}\), there exists \((A^*, S^*) \in Q_{FB}\) such that

\[ W(A) - S^{**}(A) = W(A^*) - S^*, \text{ for any } A. \]  

(A.9)

That is, the distribution of the principal’s payoff, \(W(A) - S^{**}(A)\), does not depend on the agent’s action \(A\), and we have \(U_p(W(A^*) - S(A^*)) = U_p(W^* - S^*)\), where the right-hand side is by definition the solution of the optimization in (A.4)-(A.5). The principal achieves the first best outcome regardless of the agent’s action. For the agent, because \(A^*\) is determined by (A.2), we have \(U_a(S^{**}(A^*), A^*) \geq U_a(S^{**}(A^*), A^*) \geq \bar{U}_a\). In summary, the pair \((A^*, S(A^*))\) satisfies the constraints in (A.5) and achieves the maximum value of the objective function in (A.4). It thus qualifies as a solution for the optimization problem in (A.4)-(A.5). The proof of the last claim is also straightforward, but is omitted for the sake of briefness. Q.E.D.\(^{13}\)

If the principal can ex post demand the payoff that always matches the first best outcome, she can “sell” the project to the agent in return for a “price” equal to \(W^* - A^*\). In general, however, \(W^* - A^*\) is a random variable. Indeed, when the agent is risk averse, the quantity \(W^* - A^*\) should vary with the underlying states in such a way that it achieves the optimal risk sharing between the principal and the agent at the equilibrium. Nevertheless, this contingent payoff is in no way dependent on the agent’s

\(^{13}\) As this proposition does not assume the uniqueness of the first best solution, it makes no claim about whether \((A^*, S(A^*)) = (A^*, S^*)\).
action, \( A \), and is not directly related to the project’s outcome, \( W(A) \). Therefore, in this sense, the principal-agent relation is severed at the point when the contract is signed by both parties, and it is also in this sense that we may interpret it as the principal “selling” the project to the manager. By doing so, the agent bears all the consequences of his action, and therefore the incentive problem is completely addressed. The contract can be viewed as being equivalent to the principal “selling” the project to the agent in return for a contingent payment of \( W^* - A^* \) at the terminal date. Of course, in order for it to be a feasible component of a contract, \( W^* - A^* \) cannot be contingent on ex post unverifiable conditions.

In the above analysis, we assume the manager’s utility takes the form \( U_p(S(A), A) \). It allows the manager’s action to affect the utility in a very general way. As is typical in the moral hazard literature, the manager’s utility can be specified as a function of two variables, the wage the manager receives and a private cost due to his action. That is, the utility takes the form \( U_p(S(A), c(A)) \), where \( c(A) \) is the private cost incurred by action, \( A \). We next analyze the case in which the cost of action to the manager is a constant – that is, \( c(A) \equiv c_0 \) for some constant, \( c_0 \). Under this assumption, the manager’s utility is reduced to \( U_p(S(A)) \), which is dependent on his action \( A \) only through the consequent payoff he receives. We denote the set of all feasible wealth distribution of the project by \( W(A) \) – that is, \( W(A) \equiv \{W(A) \mid A \in A\} \). We call the set \( W(A) \) a manifold if for any \( \lambda \in \mathbb{R} \) and any two attainable outcomes, \( W_1, W_2 \in W(A) \), the linear combination \( \lambda W_1 + (1-\lambda)W_2 \) is an element in \( W(A) \). We have the following proposition:

**Proposition A.2.** If, in addition to the assumptions in Proposition A.1, we assume that \( c(A) \equiv c_0 \) for some constant, \( c_0 \), and the set \( W(A) \) is a manifold, then all of the following contracts will achieve the first best outcome and therefore are optimal contracts:

\[
S(A) = S^* + \lambda(W(A) - W(A^*)), \text{ for any fixed } \lambda \in \mathbb{R}, \text{ and any } (A^*, S^*) \in Q_{FB}.
\]

**(A.10)**

**Proof.** Fix the pair \((A^*, S^*) \in Q_{FB} \). Notice that when the manager chooses action \( A^* \), we would have the payoff for the manager and that for the investor exactly match the first best solution (that is, \( S(A^*) = S^* \)). Therefore, we need only to show that, under the contract in (A.10), there is no action \( A \in A \) that the manager would prefer over \( A^* \). Assume otherwise, that there exists \( A^e \in A \) such that \( U_a(S(A^e)) > U_a(S(A^*)) \). Because \( W(A) \) is a manifold, \( \lambda W(A^e) + (1 - \lambda)W(A^*) \) can be implemented by a certain action of the manager, which we denote as \( A^\lambda \). Notice that

\[
S(A^e) = S^* + [\lambda W(A^e) + (1 - \lambda)W(A^*)] - W(A^*) = S^* + W(A^\lambda) - W(A^*) = S^e(A^\lambda)
\]

and \( S(A^e) = S^* = S^e(A^*) \). Therefore, \( U_a(S^e(A^\lambda)) = U_a(S(A^e)) > U_a(S(A^*)) = U_a(S^e(A^*)) \). This contradicts the conclusion in Proposition A.1 that \( A^* \) is the optimal strategy for the manager when facing the contract, \( S^e(A) \). Q.E.D.
The results in Propositions A.1 and A.2 generalize the results in Section I of the text.

*Derivation of the first best solution in Ou-Yang’s setting*

We state a more detailed version of the result below:

**Proposition A3.** The first best solution of the optimal portfolio policy and the optimal payment to the agent are given by

\[
A_t^* = f(t) \left[ k_t + \frac{R_a - R_p}{R_a + R_p} f^2(t) \sigma \sigma^T \right]^{-1} h, \tag{A.12}
\]

and

\[
S_T^* = \int_0^T \left( c(A_t^*, W_t^*, t) + \frac{1}{2} R_a g_t^*, g_t^* \right) dt + \int_0^T g_t^* dB_t, \tag{A.13}
\]

where \( f(t) = (1 - \gamma) e^{r(T-t)} + \gamma \), \( g_t^* = \frac{R_p}{R_a + R_p} f(t) \sigma A_t^* \), and the process \( \{W_t^*\}, \) for \( t \in [0, T] \) is given by

\[
W_t^* = e^{rT} W_0 + \int_0^t \left( e^{r(T-s)} A_s^* h \right) ds + \int_0^t \left( e^{r(T-s)} A_s^* \sigma \right) dB_s. \tag{A.14}
\]

We further have the investor’s payoff in the first best solution as

\[
W_p^* = W_T^* - S_T^* = -F + \frac{R_a}{R_a + R_p} \int_0^T f(t) A_t^* \text{diag}(P_t)^{-1} dP_t, \tag{A.15}
\]

where \( F \) is a constant.

**Proof:** Consider the portfolio manager’s expected utility,

\[
x_t = E_t \left[ U_a \left( S_T - \int_0^T c(A_t, W_t, t) dt \right) \right] < 0, \tag{A.16}
\]

which is a martingale over the Brownian fields generated by \( B_t \). A theorem of Meyer (see Jacod (1977)) holds that every martingale over the Brownian fields can be represented as an Ito stochastic integral with respect to the driftless Brownian motion: \( dx_t = \theta_t dB_t \), where \( \{\theta_t; t \geq 0\} \) is a d-dimensional adopted stochastic process and \( \int_0^T \theta_t \theta_t^T dt \) is almost surely finite. Define the process \( \{Z(t)\} \) as

\[
Z(t) = U_a^{-1}(x_t) = -\frac{1}{R_a} \ln(-R_a x_t). \tag{A.17}
\]
By Ito’s lemma, we have,
\[
Z(T) = Z(0) + \frac{1}{2} R_x \int_0^T g_t^\top g_t dt + \int_0^T g_t^\top dB_t,
\] (A.18)

where \( g_t \equiv -\frac{\theta}{R_x} g_t \). We further have that \( Z(T) = S_T - \int_0^T c(A_t, W_t, t) dt \), and from the participation constraint, \( Z(0) = Z_0 \). Hence, we get the following equivalent representation of the participation constraint:
\[
S_T = Z(T) + \int_0^T c(A_t, W_t, t) dt = Z_0 + \int_0^T c(A_t, W_t, t) dt + \frac{1}{2} R_x \int_0^T g_t^\top g_t dt + \int_0^T g_t^\top dB_t.
\] (A.19)

With the participation constraint noted in Equation (A.19), we can state the first best problem as follows:
\[
\sup_{\{A_t, g_t\}} E[U_p(N_T)]
\]
s.t.
\[
dP_t = P_t^{\mu dt + \sigma dB_t},
\]
\[
dW_t = (rW_t + A_t^h) dt + A_t^\sigma dB_t;
\]
\[
dN_t = dW_t - c(A_t, W_t, t) dt - \frac{1}{2} R_x g_t^\top g_t dt - g_t^\top dB_t,
\]

where \( N_t \equiv W_t - S_t \). Given that \( A_t \) and \( g_t \) are adapted stochastic processes, \( \{P_t; W_t; N_t\} \) are controlled Markov processes, with \( A_t \) and \( g_t \) being the controls. We define a value function process \( V(t, W_t, N_t, P_t) \) for the above optimal control problem as
\[
V(t, W_t, N_t, P_t) = \sup_{\{A_t, g_t\}} E[U_p(N_T)].
\] (A.21)

The Bellman equation for this dynamic programming problem is as follows:
\[
0 = \sup_{A, g} A^{A, g} V,
\] (A.22)

where \( A^{A, g} \) stands for the backward generating operator, i.e.,
\[ A^{a,g} V = V_t + (V_w + V_N)(rW + A^T h) + V_N \left(-c(A,W,t) - \frac{1}{2} R_a g^T g \right) + V_p \text{diag}(P_t) \mu \]

\[ + \frac{1}{2} (V_{ww} + V_{NN}) A^T \sigma \sigma^T A + \frac{1}{2} V_{NN} g^T g - V_{NN} A^T \sigma g + V_{WN} \left( A^T \sigma \sigma^T A - A^T \sigma g \right) \]

\[ + \frac{1}{2} \text{trace} \left( V_{pp} \text{diag}(P_t) \sigma \sigma^T \text{diag}(P_t) \right) + (V_{wp} + V_{np}) \text{diag}(P_t) \sigma \sigma^T A \]

\[-V_{np}^T \text{diag}(P_t) \sigma g . \]

The first-order conditions of the Bellman equation with respect to the control variables \(A\) and \(g\) are

\[ 0 = (V_w + V_N) h - V_N c_A + (V_{ww} + 2V_{wn} + V_{NN}) \sigma \sigma^T A - (V_{wn} + V_{NN}) \sigma g , \]

\[ + \sigma \sigma^T P^d (V_{wp}^T + V_{np}^T) \] (A.24)

and

\[ 0 = -V_N R_g g + V_{NN} g - (V_{NN} + V_{wn}) \sigma^T A + \sigma^T P^d V_{np}^T . \] (A.25)

Conjecture that \(V(t,W_t,N_t,P_t)\) takes the following form:

\[ V(t,W_t,N_t,P_t) = -\frac{1}{R_p} \exp \left\{ -R_p \left[ f_1(t) W_t + f_2(t) N_t + f_3(t) \right] \right\}, \] (A.26)

with the boundary conditions \(f_1(T) = 0\), and \(f_2(T) = 1\). Then we have that \(V_w = V \cdot (-R_p f_1)\), \(V_N = V \cdot (-R_p f_2)\), \(V_{ww} = V \cdot (R_p f_1)^2\), \(V_{NN} = V \cdot (R_p f_2)^2\), \(V_{wn} = V \cdot (R_p^2 f_1 f_2)\), and all the derivatives with respect to \(P\) are zeroes. Substituting all of these into the first-order conditions above, we get

\[ g_t = \frac{R_p}{R_p + R_p f_2} (f_1 + f_2) \sigma^T A , \] (A.27)

and

\[ A_t = (f_1(t) + f_2(t)) \left[ f_2(t) k_t + \frac{R_p R_p}{R_g + f_2 R_p} (f_1(t) + f_2(t))^2 \sigma \sigma^T \right]^{-1} h . \] (A.28)

To determine the values of \(f_1(t), f_2(t),\) and \(f_3(t)\), we substitute \(V\) back into the Bellman equation:

\[ 0 = W_t \left[ f_1' - (f_1 + f_2) r + f_2' \gamma \right] + N_t f_2' + f_3' - (f_1 + f_2) A_t h + f_2 \left[ A_t^T k_t A_t + \frac{1}{2} R_a g_t^T g_t \right] \]

\[ + \frac{1}{2} R_p f_2^2 g_t^T g_t + \frac{1}{2} R_p (f_1 + f_2)^2 A_t^T \sigma \sigma^T A_t - R_p f_2^2 (f_1 + f_2) A_t \sigma g_t . \] (A.29)
Notice that both $A_t$ and $g_t$ do not depend on $N_t$ or $W_t$. To eliminate the $N_t$ term from the right-hand side of the above equation, we must have $f_2' = 0$ with the boundary condition, $f_2(T) = 1$. Therefore, $f_2(t) = 1$. To eliminate the $W_t$ term, we must have

$$f_1' + f_1 r + r - \gamma = 0,$$

with the boundary condition $f_1(T) = 0$. Therefore,

$$f_1(t) = (1 - \frac{T}{T-\tau})(e^{r(T-\tau)} - 1).$$

The function $f(t)$ in theorem 1 is given by $f(t) = f_1(t) + 1$. The Bellman equation is satisfied by setting

$$f_3(s) = -\int_0^T \left( f_2 \left[ A_t^T k_t, A_t + \frac{1}{2} R_t g_t^T g_t \right] - (f_1 + f_2) A_t^h + \frac{1}{2} R_t f_2^2 g_t^T g_t \right) dt.$$

(A.32)

The following computation is not necessary for our derivation of the first best solution and for the derivation of the optimal contract. It is presented here solely for the purpose of fully replicating Equation (10) in Ou-Yang’s Theorem 1. Given the portfolio strategy $\{A_t^*\}$, the wealth process is determined by the stochastic differential equation:

$$dW_t^* = (rW_t^* + A_t^* h)dt + A_t^* \sigma dB_t.$$

(A.33)

Using Ito’s lemma, we can rewrite the equation as:

$$d(e^{r(T-t)} \cdot W_t^*) = (e^{r(T-t)} \cdot A_t^* h)dt + (e^{r(T-t)} \cdot A_t^* \sigma) dB_t.$$

(A.34)

Therefore,

$$W_t^* = e^{rT}W_0 + \int_0^T (e^{r(T-s)} \cdot A_s^* h) ds + \int_0^T (e^{r(T-s)} \cdot A_s^* \sigma) dB_s, \text{ for } t \in [0, T].$$

(A.35)

Notice that

$$S_t^* = \int_0^T \left( c(A_t^*, W_t^*, t) + \frac{1}{2} R_t g_t^* g_t^* \right) dt + \int_0^T g_t^* dB_t.$$

(A.36)

We thus have
We choose the constant, $F$, as in Ou-Yang (2003), to absorb all the constant terms in the above equation, and write the result in short as:

$$S_T^* - W_T^* = F - \frac{R_a}{R_a + R_p} \int_0^T f(t) A_t^T \text{diag}(P_t)^{-1} dP_t^r.$$  \hfill (A.38)

Hence, the proposition is proved.

**Appendix II**

**Proof of Proposition 3.** The proof is developed in two steps. First, if $\phi = 0$, the contract is the constant contract. Clearly, the agent will choose $A = 0$, which is strictly less than that in the first best. This contradicts the fact that $\phi = 0$. Thus, $\phi \neq 0$. Second, assume that $\phi < 0$. Denote the optimal contract by $C^*$. Now, consider the set of contracts: $C_\beta = \beta C^* + (1 - \beta)C^* + \alpha$, where $\alpha$ is set as before to satisfy the agent’s participation constraint. At the one extreme when we choose $\beta = 0$, we have the optimal contract with $A^* > A_{FB}$ due to the assumption that $\phi < 0$. At the other extreme when we choose $\beta = 1$, the resulting contract is the constant contract and the agent will choose $A = 0$. We can therefore choose a value of $\beta \in (0, 1)$ such that the resulting choice of $A = A_{FB}$. Notice that this choice of the contract achieves better risk sharing than contract $C^*$ because it reduces the variation in the payment to the agent. It results in a better cost-benefit tradeoff at the choice of $A_{FB}$. Therefore, $G(\beta^*) > G(1)$. This contradicts the assumption that $C^*$ is the optimal contract. Hence, we conclude that $\phi > 0$.  

\[ ... \]